

# Fractional Optimal Control in the Sense of Caputo and the Fractional Noether's Theorem<sup>1</sup>

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## Abstract

The study of fractional variational problems with derivatives in the sense of Caputo is a recent subject, the main results being Agrawal's necessary optimality conditions of Euler-Lagrange and respective transversality conditions. Using Agrawal's Euler-Lagrange equation and the Lagrange multiplier technique, we obtain here a Noether-like theorem for fractional optimal control problems in the sense of Caputo.

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## 1 Introduction

Noether's theorem, published in 1918 [23], is a central result of the calculus of variations that explains all physical laws based upon the action principle. It is a very general result, asserting that "to every variational symmetry of the problem there corresponds a conservation law". Noether's principle gives powerful insights from the various transformations that make a system invariant. For instance, in mechanics the invariance of a physical system with respect

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to spatial translation gives conservation of linear momentum; invariance with respect to rotation gives conservation of angular momentum; and invariance with respect to time translation gives conservation of energy [20, 31]. The calculus of variations is now part of a more vast discipline, called optimal control [25], and Noether's principle still holds in this more general setting [30, 32].

Fractional derivatives play an increasing role in mathematics, physics and engineering [4, 16, 17, 21, 24, 29] and the theory of the calculus of variations has been extended in order to deal with more general systems containing non-integer derivatives [1, 2, 3, 5, 9, 11, 19, 26, 27]. The new fractional variational calculus provide a more realistic approach to physics [6, 7, 8, 18, 22, 28], permitting to consider nonconservative systems in a natural way—a very important issue since closed systems do not exist: forces that do not store energy, so-called nonconservative or dissipative forces, are always present in real systems. Nonconservative forces remove energy from the systems and, as a consequence, the standard conservation laws cease to be valid. However, it is still possible to obtain a Noether-type theorem which covers both conservative and nonconservative cases [10, 12]. In a more general way, formulations of Noether's theorem were proved for fractional problems with left and right Riemann-Liouville fractional derivatives [9, 13, 14, 15].

In [3] Agrawal proves a version of the Euler-Lagrange equations for fractional problems of the calculus of variations in the sense of Caputo. One of the interesting aspects of the new theory is that both Caputo and Riemann-Liouville derivatives play a role in Agrawal's Euler-Lagrange equations. Here we use the results of [3] to formulate a Noether-type theorem in the general context of the fractional optimal control in the sense of Caputo.

## 2 Fractional Derivatives

In this section we collect the well-known definitions of fractional derivatives in the sense of Riemann-Liouville and Caputo (see [1, 3, 21, 24, 29]).

**Definition 2.1** (Fractional derivative in the sense of Riemann-Liouville). *Let  $f$  be an integrable continuous function in the interval  $[a, b]$ . For  $t \in [a, b]$ , the left Riemann-Liouville fractional derivative  ${}_a D_t^\alpha f(t)$  and the right Riemann-Liouville fractional derivative  ${}_t D_b^\alpha f(t)$ , of order  $\alpha$ , are defined by*

$${}_a D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_a^t (t-\theta)^{n-\alpha-1} f(\theta) d\theta, \quad (1)$$

$${}_t D_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left( -\frac{d}{dt} \right)^n \int_t^b (\theta-t)^{n-\alpha-1} f(\theta) d\theta, \quad (2)$$

where  $n \in \mathbb{N}$ ,  $n-1 \leq \alpha < n$ , and  $\Gamma$  is the Euler gamma function.

**Definition 2.2** (Fractional derivative in the sense of Caputo). *Let  $f$  be an integrable continuous function in  $[a, b]$ . For  $t \in [a, b]$ , the left Caputo fractional derivative  ${}_a^C D_t^\alpha f(t)$  and the right Caputo fractional derivative  ${}_t^C D_b^\alpha f(t)$ , of order  $\alpha$ , are defined in the following way:*

$${}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\theta)^{n-\alpha-1} \left( \frac{d}{d\theta} \right)^n f(\theta) d\theta, \quad (3)$$

$${}_t^C D_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_t^b (\theta-t)^{n-\alpha-1} \left( -\frac{d}{d\theta} \right)^n f(\theta) d\theta, \quad (4)$$

where  $n \in \mathbb{N}$ ,  $n-1 \leq \alpha < n$ .

**Remark 2.3.** If  $\alpha \in \mathbb{N}$ , equalities (1)-(4) give the classical derivatives

$$\begin{aligned} {}_a D_t^\alpha f(t) &= {}_a^C D_t^\alpha f(t) = \left( \frac{d}{dt} \right)^\alpha f(t), \\ {}_t D_b^\alpha f(t) &= {}_t^C D_b^\alpha f(t) = \left( -\frac{d}{dt} \right)^\alpha f(t). \end{aligned}$$

**Remark 2.4.** The Caputo fractional derivative of a constant is always equal to zero. This is not the case with the Riemann-Liouville fractional derivative.

### 3 Main Results

Our main result is a Noether-type theorem for fractional optimal control problems in the sense of Caputo (Theorem 3.17). As a corollary, we obtain a Noether theorem for the fractional problems of the calculus of variations (Corollary 3.20).

The fractional optimal control problem in the sense of Caputo is introduced, without loss of generality, in Lagrange form:

$$\begin{aligned} I[q(\cdot), u(\cdot)] &= \int_a^b L(t, q(t), u(t)) dt \longrightarrow \min, \\ {}_a^C D_t^\alpha q(t) &= \varphi(t, q(t), u(t)), \end{aligned} \quad (P_C)$$

where functions  $q : [a, b] \rightarrow \mathbb{R}^n$  satisfy appropriate boundary conditions. The Lagrangian  $L : [a, b] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$  and the velocity vector  $\varphi : [a, b] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  are assumed to be functions of class  $C^1$  with respect to all their arguments. We also assume, without loss of generality, that  $0 < \alpha \leq 1$ . In conformity with the calculus of variations, we are considering that the control functions  $u(\cdot)$  take values on an open set of  $\mathbb{R}^m$ . Along the work we denote by  $\partial_i L$ ,  $i = 1, 2, 3$ , the partial derivative of function  $L(\cdot, \cdot, \cdot)$  with respect to its  $i$ th argument.

**Definition 3.1** (Process). *An admissible pair  $(q(\cdot), u(\cdot))$  which satisfies the control system  ${}_a^C D_t^\alpha q(t) = \varphi(t, q(t), u(t))$  of problem  $(P_C)$  is said to be a process.*

**Remark 3.2.** *Choosing  $\alpha = 1$ , Problema  $(P_C)$  is reduced to the classical problem of optimal control theory [25]:*

$$\begin{aligned} I[q(\cdot), u(\cdot)] &= \int_a^b L(t, q(t), u(t)) dt \longrightarrow \min, \\ \dot{q}(t) &= \varphi(t, q(t), u(t)). \end{aligned} \quad (5)$$

**Remark 3.3.** *The fundamental fractional problem of the calculus of variations in the sense of Caputo, first introduced in [3]*

$$I[q(\cdot)] = \int_a^b L(t, q(t), {}_a^C D_t^\alpha q(t)) dt \longrightarrow \min, \quad (6)$$

is a particular case of  $(P_C)$ : we just need to choose  $\varphi(t, q, u) = u$ .

The fractional Hamiltonian formalism introduced in [11] is easily adapted to our present context. Using the standard Lagrange multiplier technique, we rewrite problem  $(P_C)$  in the following equivalent form:

$$I[q(\cdot), u(\cdot), p(\cdot)] = \int_a^b [\mathcal{H}(t, q(t), u(t), p(t)) - p(t) \cdot {}_a^C D_t^\alpha q(t)] dt \longrightarrow \min, \quad (7)$$

where the Hamiltonian  $\mathcal{H}$  is defined by

$$\mathcal{H}(t, q, u, p) = L(t, q, u) + p \cdot \varphi(t, q, u). \quad (8)$$

**Remark 3.4.** *In the context of classical mechanics,  $p$  is interpreted as the generalized momentum. In the optimal control literature, the multiplier  $p$  is known as the adjoint variable.*

We now proceed with the usual steps for obtaining necessary optimality conditions in the calculus of variations. We begin by computing the variation  $\delta I$  of functional (7):

$$\delta I = \int_a^b [\partial_2 \mathcal{H} \cdot \delta q + \partial_3 \mathcal{H} \cdot \delta u + \partial_4 \mathcal{H} \cdot \delta p - \delta p \cdot {}_a^C D_t^\alpha q - p \cdot \delta ({}_a^C D_t^\alpha q)] dt, \quad (9)$$

where  $\delta q$ ,  $\delta u$  and  $\delta p$  are the variations of  $q$ ,  $u$ , and  $p$  respectively. Using the results obtained in [1, 2, 3], equation (9) is equivalent to

$$\begin{aligned} \delta I &= \int_a^b [(\partial_2 \mathcal{H} - {}_t D_b^\alpha p) \cdot \delta q + \partial_3 \mathcal{H} \cdot \delta u + (\partial_4 \mathcal{H} - {}_a^C D_t^\alpha q) \cdot \delta p] dt \\ &\quad - ({}_t D_b^{\alpha-1} p) \cdot \delta q|_a^b \end{aligned}$$

where the fractional derivatives of  $p(t)$  are in the sense of Riemann-Liouville (in contrast with the fractional derivative of  $q(t)$  which is taken in the sense of Caputo). Standard arguments conduce us to the following result.

**Theorem 3.5.** *If  $(q(\cdot), u(\cdot))$  is an optimal process for problem  $(P_C)$ , then there exists a function  $p(\cdot) \in C^1([a, b]; \mathbb{R}^n)$  such that for all  $t \in [a, b]$  the tuple  $(q(\cdot), u(\cdot), p(\cdot))$  satisfy the following conditions:*

- the Hamiltonian system

$$\begin{cases} \partial_2 \mathcal{H}(t, q(t), u(t), p(t)) = {}_t D_b^\alpha p(t), \\ \partial_4 \mathcal{H}(t, q(t), u(t), p(t)) = {}_a^C D_t^\alpha q(t); \end{cases}$$

- the stationary condition

$$\partial_3 \mathcal{H}(t, q(t), u(t), p(t)) = 0;$$

- the transversality condition

$$({}_t D_b^{\alpha-1} p) \cdot \delta q|_a^b = 0;$$

with  $\mathcal{H}$  given by (8).

**Definition 3.6.** *A triple  $(q(\cdot), u(\cdot), p(\cdot))$  satisfying Theorem 3.5 will be called a fractional Pontryagin extremal.*

**Remark 3.7.** *For the fundamental fractional problem of the calculus of variations in the sense of Caputo (6) we have  $\mathcal{H} = L + p \cdot u$ . It follows from Theorem 3.5 that*

$$\begin{aligned} {}_a^C D_t^\alpha q &= u, \\ {}_t D_b^\alpha p &= \partial_2 L, \\ \partial_3 \mathcal{H} = 0 &\Leftrightarrow p = -\partial_3 L \Rightarrow {}_t D_b^\alpha p = -{}_t D_b^\alpha \partial_3 L. \end{aligned} \tag{10}$$

Comparing both expressions for  ${}_t D_b^\alpha p$ , we arrive to the fractional Euler-Lagrange equations proved by O. P. Agrawal [3, §3]:

$$\partial_2 L + {}_t D_b^\alpha \partial_3 L = 0. \tag{11}$$

In other words, for the problem of the calculus of variations (6) the fractional Pontryagin extremals give Agrawal's Euler-Lagrange extremals.

**Remark 3.8.** Our optimal control problem  $(P_C)$  only involves Caputo fractional derivatives but both Caputo and Riemann-Liouville fractional derivatives appear in the necessary optimality condition given by Theorem 3.5. This is different from [2, 14, 15] where the necessary conditions only involve the same type of derivatives (Riemann-Liouville) as those in the definition of the fractional optimal control problem. This fact also occurs in the particular case of the calculus of variations as noted in [3]: the Riemann-Liouville derivative is present in the necessary condition of optimality (11) but not in the formulation of the problem (6).

The notion of variational invariance for problem  $(P_C)$  is defined with the help of the equivalent problem (7).

**Definition 3.9** (Invariance of  $(P_C)$  without transformation of time). We say that functional  $(P_C)$  is invariant under the one-parameter family of infinitesimal transformations

$$\begin{cases} \bar{q}(t) = q(t) + \varepsilon \xi(t, q, u, p) + o(\varepsilon), \\ \bar{u}(t) = u(t) + \varepsilon \varsigma(t, q, u, p) + o(\varepsilon), \\ \bar{p}(t) = p(t) + \varepsilon \varrho(t, q, u, p) + o(\varepsilon), \end{cases}$$

if and only if

$$\begin{aligned} & \int_{t_a}^{t_b} [\mathcal{H}(t, q(t), u(t), p(t)) - p(t) \cdot {}_a^C D_t^\alpha q(t)] dt \\ &= \int_{t_a}^{t_b} [\mathcal{H}(t, \bar{q}(t), \bar{u}(t), \bar{p}(t)) - \bar{p}(t) \cdot {}_a^C D_t^\alpha \bar{q}(t)] dt \quad (12) \end{aligned}$$

for any subinterval  $[t_a, t_b] \subseteq [a, b]$ .

Noether's theorem will be proved following similar steps as those in [13] for the problems of the calculus of variations in the Riemann-Liouville sense.

**Lemma 3.10** (necessary and sufficient condition of invariance). If functional  $(P_C)$  is invariant, in the sense of Definition 3.9, then

$$\partial_2 \mathcal{H}(t, q, u, p) \cdot \xi + \partial_3 \mathcal{H}(t, q, u, p) \cdot \varsigma + (\partial_4 \mathcal{H}(t, q, u, p) - {}_a^C D_t^\alpha q) \cdot \varrho - p \cdot {}_a^C D_t^\alpha \xi = 0. \quad (13)$$

*Proof.* Since condition (12) is to be valid for any subinterval  $[t_a, t_b] \subseteq [a, b]$ , we can write (12) in the following equivalent form:

$$\mathcal{H}(t, q(t), u(t), p(t)) - p(t) \cdot {}_a^C D_t^\alpha q(t) = \mathcal{H}(t, \bar{q}(t), \bar{u}(t), \bar{p}(t)) - \bar{p}(t) \cdot {}_a^C D_t^\alpha \bar{q}(t). \quad (14)$$

We differentiate both sides of (14) with respect to  $\varepsilon$  and then substitute  $\varepsilon$  by zero. The definition and properties of the Caputo fractional derivative permit us to write that

$$\begin{aligned}
0 &= \partial_2 \mathcal{H}(t, q, u, p) \cdot \xi + \partial_3 \mathcal{H}(t, q, u, p) \cdot \varsigma + (\partial_4 \mathcal{H}(t, q, u, p) - {}_a^C D_t^\alpha q) \cdot \varrho \\
&\quad - p \cdot \frac{d}{d\varepsilon} \left[ \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\theta)^{n-\alpha-1} \left( \frac{d}{d\theta} \right)^n q(\theta) d\theta \right. \\
&\quad \left. + \frac{\varepsilon}{\Gamma(n-\alpha)} \int_a^t (t-\theta)^{n-\alpha-1} \left( \frac{d}{d\theta} \right)^n \xi d\theta \right]_{\varepsilon=0} \\
&= \partial_2 \mathcal{H}(t, q, u, p) \cdot \xi(t, q) + \partial_3 \mathcal{H}(t, q, u, p) \cdot \varsigma + (\partial_4 \mathcal{H}(t, q, u, p) - {}_a^C D_t^\alpha q) \cdot \varrho \\
&\quad - p \cdot \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\theta)^{n-\alpha-1} \left( \frac{d}{d\theta} \right)^n \xi d\theta. \quad (15)
\end{aligned}$$

Expression (15) is equivalent to (13).  $\square$

In this work we propose the new notion of *fractional conservation law in the sense of Caputo*. For that, we introduce the operator  $\mathcal{D}_t^\omega$ .

**Definition 3.11.** *Given two functions  $f$  and  $g$  of class  $C^1$  in the interval  $[a, b]$ , we introduce the following operator:*

$$\mathcal{D}_t^\omega [f, g] = -g {}_t D_b^\omega f + f {}_a^C D_t^\omega g,$$

where  $t \in [a, b]$  and  $\omega \in \mathbb{R}_0^+$ .

**Remark 3.12.** *Similar operators were used in [11] and [13, Def. 19]. While here  $\mathcal{D}_t^\omega$  depends both on Caputo and Riemann-Liouville derivatives, the ones introduced in [11, 13] involve Riemann-Liouville derivatives only. We note that in the classical context  $\omega = 1$ , and  $\mathcal{D}_t^1[f, g] = gf' + fg' = \frac{d}{dt}(fg) = \mathcal{D}_t^1[g, f]$ .*

**Definition 3.13** (fractional conservation law in the sense of Caputo – cf. Def. 23 of [13]). *A quantity  $C_f(t, q(t), {}_a^C D_t^\alpha q(t), u(t), p(t))$  is said to be a fractional conservation law in the sense of Caputo if it is possible to write  $C_f$  as a sum of products,*

$$\begin{aligned}
C_f(t, q(t), {}_a^C D_t^\alpha q(t), u(t), p(t)) \\
= \sum_{i=1}^r C_i^1(t, q(t), {}_a^C D_t^\alpha q(t), u(t), p(t)) \cdot C_i^2(t, q(t), {}_a^C D_t^\alpha q(t), u(t), p(t)) \quad (16)
\end{aligned}$$

for some  $r \in \mathbb{N}$ , and for every  $i = 1, \dots, r$  the pair  $C_i^1$  and  $C_i^2$  satisfy one of the following relations:

$$\mathcal{D}_t^\alpha [C_i^1(t, q(t), {}_a^C D_t^\alpha q(t), u(t), p(t)), C_i^2(t, q(t), {}_a^C D_t^\alpha q(t), u(t), p(t))] = 0 \quad (17)$$

or

$$\mathcal{D}_t^\alpha [C_i^2(t, q(t), {}_a^C D_t^\alpha q(t), u(t), p(t)), C_i^1(t, q(t), {}_a^C D_t^\alpha q(t), u(t), p(t))] = 0 \quad (18)$$

along all the fractional Pontryagin extremals (Definition 3.6).

**Remark 3.14.** If  $\alpha = 1$  (17) and (18) coincide, and  $C_f$  (16) satisfy the classical definition of conservation law:  $\frac{d}{dt} [C_f(t, q, \dot{q})] = 0$ .

**Lemma 3.15** (Noether's theorem without transformation of time). *If functional  $(P_C)$  is invariant in the sense of Definition 3.9, then*

$$p(t) \cdot \xi$$

is a fractional conservation law in the sense of Caputo.

*Proof.* We use the conditions of Theorem 3.5 in the necessary and sufficient condition of invariance (13):

$$\begin{aligned} 0 &= -\partial_2 \mathcal{H} \cdot \xi - \partial_3 \mathcal{H} \cdot \varsigma - (\partial_4 \mathcal{H} - {}_a^C D_t^\alpha q) \cdot \varrho + p \cdot {}_a^C D_t^\alpha \xi \\ &= -\xi \cdot {}_t D_b^\alpha p + p \cdot {}_a^C D_t^\alpha \xi \\ &= \mathcal{D}_t^\alpha [p, \xi]. \end{aligned}$$

□

**Definition 3.16** (Invariance of  $(P_C)$ ). *Functional  $(P_C)$  is said to be invariant under the one-parameter infinitesimal transformations*

$$\begin{cases} \bar{t} = t + \varepsilon \tau(t, q, u, p) + o(\varepsilon), \\ \bar{q}(t) = q(t) + \varepsilon \xi(t, q, u, p) + o(\varepsilon), \\ \bar{u}(t) = u(t) + \varepsilon \varsigma(t, q, u, p) + o(\varepsilon), \\ \bar{p}(t) = p(t) + \varepsilon \varrho(t, q, u, p) + o(\varepsilon), \end{cases} \quad (19)$$

if and only if

$$\begin{aligned} &\int_{t_a}^{t_b} [\mathcal{H}(t, q(t), u(t), p(t)) - p(t) \cdot {}_a^C D_t^\alpha q(t)] dt \\ &= \int_{\bar{t}(t_a)}^{\bar{t}(t_b)} [\mathcal{H}(\bar{t}, \bar{q}(\bar{t}), \bar{u}(\bar{t}), \bar{p}(\bar{t})) - \bar{p}(\bar{t}) \cdot {}_a^C D_{\bar{t}}^\alpha \bar{q}(\bar{t})] d\bar{t} \quad (20) \end{aligned}$$

for any subinterval  $[t_a, t_b] \subseteq [a, b]$ .

Next result provides an extension of Noether's theorem [30] for fractional optimal control problems in the sense of Caputo.

**Theorem 3.17** (Noether's theorem for fractional optimal control problems). *If the functional  $(P_C)$  is invariant under the one-parameter infinitesimal transformations (19), then*

$$\begin{aligned} C_f(t, q(t), {}_a^C D_t^\alpha q(t), u(t), p(t)) \\ = [\mathcal{H}(t, q(t), u(t), p(t)) - (1 - \alpha)p(t) \cdot {}_a^C D_t^\alpha q(t)] \tau - p(t) \cdot \xi \end{aligned} \quad (21)$$

is a fractional conservation law in the sense of Caputo (see Definition 3.13).

**Remark 3.18.** If  $\alpha = 1$  problem  $(P_C)$  takes the classical form (5) and Theorem 3.17 gives the conservation law of [30]:

$$C(t, q(t), u(t), p(t)) = [\mathcal{H}(t, q(t), u(t), p(t))] \tau - p(t) \cdot \xi.$$

*Proof.* Every non-autonomous problem  $(P_C)$  is equivalent to an autonomous one by artificially considering  $t$  as a dependent variable. For that we consider a Lipschitzian transformation

$$[a, b] \ni t \longmapsto \sigma f(\lambda) \in [\sigma_a, \sigma_b]$$

satisfying the condition  $t'_\sigma = \frac{dt(\sigma)}{d\sigma} = f(\lambda) = 1$  for  $\lambda = 0$ , such that (7) takes the form

$$\begin{aligned} \bar{I}[t(\cdot), q(t(\cdot)), u(t(\cdot)), p(t(\cdot))] &= \int_{\sigma_a}^{\sigma_b} [\mathcal{H}(t(\sigma), q(t(\sigma)), u(t(\sigma)), p(t(\sigma)) \\ &\quad - p(t(\sigma)) \cdot {}_{\sigma_a}^C D_{t(\sigma)}^\alpha q(t(\sigma))] t'_\sigma d\sigma, \end{aligned}$$

where  $t(\sigma_a) = a$ ,  $t(\sigma_b) = b$  and

$$\begin{aligned} {}_{\sigma_a}^C D_{t(\sigma)}^\alpha q(t(\sigma)) \\ &= \frac{1}{\Gamma(n - \alpha)} \int_{\frac{a}{f(\lambda)}}^{\sigma f(\lambda)} (\sigma f(\lambda) - \theta)^{n-\alpha-1} \left( \frac{d}{d\theta} \right)^n q(\theta f^{-1}(\lambda)) d\theta \\ &= \frac{(t'_\sigma)^{-\alpha}}{\Gamma(n - \alpha)} \int_{\frac{a}{(t'_\sigma)^2}}^{\sigma} (\sigma - s)^{n-\alpha-1} \left( \frac{d}{ds} \right)^n q(s) ds \\ &= (t'_\sigma)^{-\alpha} {}_{\chi}^C D_\sigma^\alpha q(\sigma), \quad \left( \chi = \frac{a}{(t'_\sigma)^2} \right). \end{aligned}$$

Then, we have

$$\begin{aligned} \bar{I}[t(\cdot), q(t(\cdot)), u(t(\cdot)), p(t(\cdot))] \\ &= \int_{\sigma_a}^{\sigma_b} [\mathcal{H}(t(\sigma), q(t(\sigma)), u(t(\sigma)), p(t(\sigma)) - p(t(\sigma)) \cdot (t'_\sigma)^{-\alpha} {}_{\chi}^C D_\sigma^\alpha q(\sigma)] t'_\sigma d\sigma \\ &\doteq \int_{\sigma_a}^{\sigma_b} \bar{\mathcal{H}}_f(t(\sigma), t'_\sigma, q(t(\sigma)), u(t(\sigma)), p(t(\sigma)), {}_{\chi}^C D_\sigma^\alpha q(\sigma)) d\sigma \\ &= \int_a^b [\mathcal{H}(t, q(t), u(t), p(t)) - p(t) \cdot {}_a^C D_t^\alpha q(t)] dt \\ &= I[q(\cdot), u(\cdot), p(\cdot)]. \end{aligned}$$

If functional  $I[q(\cdot), u(\cdot), p(\cdot)]$  is invariant in the sense of Definition 3.16, then the functional  $\bar{I}[t(\cdot), q(t(\cdot)), u(t(\cdot)), p(t(\cdot))]$  is invariant in the sense of Definition 3.9. Applying Lemma 3.15, we obtain that

$$C_f \left( t(\sigma), t'_\sigma, q(t(\sigma)), u(t(\sigma)), p(t(\sigma)), {}_x^C D_\sigma^\alpha q(\sigma) \right) = p(t(\sigma)) \cdot \xi + \psi(t(\sigma)) \tau \quad (22)$$

is a fractional conservation law in the sense of Caputo. For  $\lambda = 0$ ,

$$p(t(\sigma)) = p(t) \quad (23)$$

and it follows from the stationary condition of Theorem 3.5 (see third equality in (10)) that

$$\begin{aligned} \psi &= - \frac{\partial \bar{H}_f}{\partial t'_\sigma} = \frac{\partial}{\partial t'_\sigma} \left[ p(t(\sigma)) \cdot \frac{(t'_\sigma)^{-\alpha}}{\Gamma(n-\alpha)} \int_{\frac{a}{(t'_\sigma)^2}}^\sigma (\sigma-s)^{n-\alpha-1} \left( \frac{d}{ds} \right)^n q(s) ds \right] t'_\sigma \\ &\quad - \mathcal{H} + p \cdot {}_a^C D_t^\alpha q \\ &= - \alpha p(t(\sigma)) \cdot \frac{(t'_\sigma)^{-\alpha-1}}{\Gamma(n-\alpha)} \int_{\frac{a}{(t'_\sigma)^2}}^\sigma (\sigma-s)^{n-\alpha-1} \left( \frac{d}{ds} \right)^n q(s) ds - \mathcal{H} + p \cdot {}_a^C D_t^\alpha q \\ &= - (\mathcal{H} - (1-\alpha)p \cdot {}_a^C D_t^\alpha q) . \end{aligned} \quad (24)$$

Substituting (23) and (24) into (22), we obtain the fractional conservation law (21).  $\square$

As a corollary, we obtain the analogous to the main result proved in [13] for fractional problems of the calculus of variations in the Riemann-Liouville sense.

**Definition 3.19** (variational invariance for (6)). *Functional (6) is said to be invariant under the one-parameter family of infinitesimal transformations*

$$\begin{cases} \bar{t} = t + \varepsilon \tau(t, q) + o(\varepsilon), \\ \bar{q}(t) = q(t) + \varepsilon \xi(t, q) + o(\varepsilon), \end{cases} \quad (25)$$

*if and only if*

$$\int_{t_a}^{t_b} L(t, q(t), {}_a^C D_t^\alpha q(t)) dt = \int_{\bar{t}(t_a)}^{\bar{t}(t_b)} L(\bar{t}, \bar{q}(\bar{t}), {}_a^C D_{\bar{t}}^\alpha \bar{q}(\bar{t}),) d\bar{t}$$

*for any subinterval  $[t_a, t_b] \subseteq [a, b]$ .*

**Corollary 3.20** (Noether's theorem for fractional problems of the calculus of variations). *If functional (6) is invariant under the family of transformations (25), then*

$$C_f(t, q, {}_a^C D_t^\alpha q) = \partial_3 L(t, q, {}_a^C D_t^\alpha q) \cdot \xi + [L(t, q, {}_a^C D_t^\alpha q) - \alpha \partial_3 L(t, q, {}_a^C D_t^\alpha q) \cdot {}_a^C D_t^\alpha q] \tau \quad (26)$$

is a fractional conservation law in the sense of Caputo.

*Proof.* The fractional conservation law (26) is obtained applying Theorem 3.17 to functional (6).  $\square$

**Remark 3.21.** *If  $\alpha = 1$  problem (6) is reduced to the classical problem of the calculus of variations,*

$$I[q(\cdot)] = \int_a^b L(t, q(t), \dot{q}(t)) \longrightarrow \min, \quad (27)$$

and one obtains from Corollary 3.20 the standard Noether's theorem [23]:

$$C(t, q, \dot{q}) = \partial_3 L(t, q, \dot{q}) \cdot \xi(t, q) + [L(t, q, \dot{q}) - \partial_3 L(t, q, \dot{q}) \cdot \dot{q}] \tau(t, q) \quad (28)$$

is a conservation law, i.e. (28) is constant along all the solutions of the Euler-Lagrange equations

$$\partial_2 L(t, q, \dot{q}) = \frac{d}{dt} \partial_3 L(t, q, \dot{q}) \quad (29)$$

(these classical equations are obtained from (11) putting  $\alpha = 1$ ).

## 4 Illustrative Examples

In classical mechanics, when problem (27) does not depend explicitly on  $q$ , i.e.  $L = L(t, \dot{q})$ , it follows from (10) and (29) that the generalized momentum  $p$  is a conservation law. This is also an immediate consequence of Noether's theorem [23]: from the invariance with respect to translations on  $q$  ( $\tau = 0, \xi = 1$ ), it follows from (28) that  $p = \partial_3 L$  is a conservation law. Another famous example of application of Noether's theorem in classical mechanics is given by the conservation of energy: when the Lagrangian  $L$  in (27) is autonomous, i.e.  $L = L(q, \dot{q})$ , we have invariance under time-translations ( $\tau = 1, \xi = 0$ ) and it follows from (8), (10) and (28) that the Hamiltonian  $\mathcal{H}$  (which is interpreted as being the energy in classical mechanics) is a conservation law. Surprisingly enough, we show next, as an immediate consequence of our Theorem 3.17, that for the problem  $(P_C)$  with a fractional order of differentiation  $\alpha$  ( $\alpha \neq 1$ ), the following happens:

- (i) similarly to classical mechanics, the generalized momentum  $p$  is a fractional conservation law when  $L$  and  $\varphi$  do not depend explicitly on  $q$  (Example 4.1);
- (ii) differently from classical mechanics, the Hamiltonian  $\mathcal{H}$  is not a fractional conservation law when  $L$  and  $\varphi$  are autonomous (Example 4.2).

In situation (ii), we obtain from our Theorem 3.17 a new fractional conservation law that involves not only the Hamiltonian  $\mathcal{H}$  but also the fractional order of differentiation  $\alpha$ , the generalized momentum  $p$ , and the Caputo derivative of the state trajectory  $q$  (see (30) below). This is in agreement with the claim that the fractional calculus of variations provide a very good formalism to model nonconservative mechanics [11, 27]. In the classical case we have  $\alpha = 1$  and the new obtained fractional conservation law (30) reduces to the expected “conservation of energy”  $\mathcal{H}$ .

**Example 4.1.** Let us consider problem  $(P_C)$  with  $L(t, q, u) = L(t, u)$ ,  $\varphi(t, q, u) = \varphi(t, u)$ . Such a problem is invariant under translations on the variable  $q$ , i.e. condition (20) is verified for  $\bar{t} = t$ ,  $\bar{q}(t) = q(t) + \varepsilon$ ,  $\bar{u}(\bar{t}) = u(t)$  and  $\bar{p}(\bar{t}) = p(t)$ : we have  $d\bar{t} = dt$  and condition (20) is satisfied since  ${}_a^C D_t^\alpha \bar{q}(\bar{t}) = {}_a^C D_t^\alpha q(t)$ :

$$\begin{aligned} {}_a^C D_t^\alpha \bar{q}(\bar{t}) &= \frac{1}{\Gamma(n-\alpha)} \int_{\bar{a}}^{\bar{t}} (\bar{t}-\theta)^{n-\alpha-1} \left( \frac{d}{d\theta} \right)^n \bar{q}(\theta) d\theta \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\theta)^{n-\alpha-1} \left( \frac{d}{d\theta} \right)^n (q(t) + \varepsilon) d\theta \\ &= {}_a^C D_t^\alpha q(t) + {}_a^C D_t^\alpha \varepsilon \\ &= {}_a^C D_t^\alpha q(t). \end{aligned}$$

According with (19) one has  $\xi = 1$  and  $\tau = \varsigma = \varrho = 0$ . It follows from Theorem 3.17 that  $p(t)$  is a fractional conservation law in the sense of Caputo.

**Example 4.2.** We now consider the autonomous problem  $(P_C)$ :  $L(t, q, u) = L(q, u)$  and  $\varphi(t, q, u) = \varphi(q, u)$ . This problem is invariant under time translation, i.e. the invariance condition (20) is verified for  $\bar{t} = t + \varepsilon$ ,  $\bar{q}(\bar{t}) = q(t)$ ,  $\bar{u}(\bar{t}) = u(t)$  and  $\bar{p}(\bar{t}) = p(t)$ : we have  $d\bar{t} = dt$  and (20) follows from the fact

that  ${}_a^C D_t^\alpha \bar{q}(\bar{t}) = {}_a^C D_t^\alpha q(t)$ :

$$\begin{aligned}
 {}_a^C D_t^\alpha \bar{q}(\bar{t}) &= \frac{1}{\Gamma(n-\alpha)} \int_{\bar{a}}^{\bar{t}} (\bar{t}-\theta)^{n-\alpha-1} \left( \frac{d}{d\theta} \right)^n \bar{q}(\theta) d\theta \\
 &= \frac{1}{\Gamma(n-\alpha)} \int_{a+\varepsilon}^{t+\varepsilon} (t+\varepsilon-\theta)^{n-\alpha-1} \left( \frac{d}{d\theta} \right)^n \bar{q}(\theta) d\theta \\
 &= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} \left( \frac{d}{ds} \right)^n \bar{q}(t+\varepsilon) ds \\
 &= {}_a^C D_t^\alpha \bar{q}(t+\varepsilon) = {}_a^C D_t^\alpha \bar{q}(\bar{t}) \\
 &= {}_a^C D_t^\alpha q(t).
 \end{aligned}$$

With the notation (19) one has  $\tau = 1$  and  $\xi = \varsigma = \varrho = 0$ . We conclude from Theorem 3.17 that

$$\mathcal{H}(t, q, u, p) - (1-\alpha)p \cdot {}_a^C D_t^\alpha q \quad (30)$$

is a fractional conservation law in the sense of Caputo. For  $\alpha = 1$  (30) represents the “conservation of the total energy”:

$$\mathcal{H}(t, q(t), u(t), p(t)) = \text{constant}, \quad t \in [a, b],$$

for any Pontryagin extremal  $(q(\cdot), u(\cdot), p(\cdot))$  of the problem.

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